Segments of Stable Polynomials, Local Convex Directions and the Minimum Left Extreme

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Abstract

In this paper, we proposed a matrix inequality for testing the stability of the segment of polynomials determined by p_0 and p_1, where p_0 and p_1 are Hurwitz stable polynomials. Furthermore, if p_0 is a stable polynomial we obtain local convex directions with respect to p_0 using the solutions of this inequality. Finally given the stable polynomial p_0 and the coefficients of p_1 satisfying the same matrix inequality, we find a number k_0 such that $p_0(t) + kp_1(t)$ is Hurwitz for every $k > k_0$, that means estimating the minimum left extreme, which was studied by Bialas. **Key words:** Hurwitz polynomials, segments of polynomials, matrix inequality, matrices of monotone kind.

1 Introduction

Motivated by the robustness analysis of systems with uncertain parameters, different approaches to study the stability of segment of polynomials have been proposed ([4], [5], [7], [8], [13]). The question is to find conditions on the stable polynomials $p_0(t)$ and $p_1(t)$ with $n = \deg(p_0) \geq \deg(p_1)$ such that the segment of polynomials described by $P(t, \lambda) = \lambda p_0(t) + (1 - \lambda)p_1(t)$ is stable for all $\lambda \in [0, 1]$. There are several important results where necessary and sufficient conditions were obtained: Bialas's Theorem (see [2] and [4]), the Segment Lemma, which was stablished by Chapellat and Bhattacharyya (see [3] and [8]), and the Bose's test [6].

Besides the three necessary and sufficient conditions mentioned above several algorithms have been developed to test efficiently the stability of segments of polynomials: using the Segment Lemma there was developed an algorithm in [7]. Sufficients conditions to prove the stability of segments of polynomials were gotten by Rantzer (see [10] and [13]). Conditions in the frequency domain were derived in [5]. Recently, in [11] it was obtained a procedure to check the Hurwitz stability of convex combinations of polynomials in a finite number of operations. Following the ideas exposed in [1] this paper address the problem of obtaining simple algebraic conditions for checking the stability of a segment of polynomials. Although our approach provides a sufficient condition, has the advantage to be useful when $\deg(p_0) = n$ and $\deg(p_1) = n, n-1$ in contrast with the Bialas's Theorem where it is supposed that $\deg(p_0) > \deg(p_1)$, or the Segment Lemma and Bose's Test where it is supposed that $\deg(p_0) = \deg(p_1)$.

In the above mentioned works ([4], [5], [7], [8], [13]) it is supposed that the two extrems are stable or one is stable and the other one is semistable. Unlike such works, our approach can be applied, supposing at the beginning that only one is stable. The situation is that we know only one stable polynomial $p_0(t)$ and the problem is to find polynomials $p_1(t)$ such that $\lambda p_0(t) + (1 - \lambda)p_1(t)$ is Hurwitz for all $\lambda \in [0, 1]$. In this case we say that $p_1 - p_0$ is a local convex di-

ISBN: 970-32-2137-8

rection to p_0. We find some of such polynomials p_1. Our approach is as follows.

Consider a stable polynomial $p_0(t) = t^n + a_1 t^{n-1} + ... + a_n$ which is the nominal polynomial and let $p_1(t)$ be an arbitrary polynomial of degree $n-1$. Define the matrix

$$E_{(n,n-1)} = \begin{pmatrix} a_1 & -1 & 0 & 0 & ... & 0 & 0 \\ -a_3 & a_2 & -a_1 & 1 & ... & 0 & 0 \\ a_5 & -a_4 & a_3 & -a_2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & a_{n-1} & -a_{n-2} \\ 0 & 0 & 0 & 0 & ... & 0 & a_n \end{pmatrix} \quad (1)$$

If the polynomials $p_0(t) = t^n + a_1 t^{n-1} + ... + a_n$ and $p_1(t) = c_1 t^{n-1} + c_2 t^{n-2} + ... + c_n$ are Hurwitz and the vector $c = (c_1, c_2, ..., c_n)^T \succeq 0$ satisfies the system of linear inequalities

$$E_{(n,n-1)} c \gneq 0 \quad (2)$$

then, the convex combination $\lambda p_0(t) + (1-\lambda)p_1(t)$ is Hurwitz for every $\lambda \in [0,1]$. Here the symbol $\succeq 0$ ($\preceq 0$) means that every component of a given vector is nonnegative (nonpositive) and the symbol $\gneq 0$ means that every component of a given vector is nonnegative but there is a component different from zero.

Besides we study the solutions of (2), which means to find local convex directions, $p_1 - p_0$, with respect p_0. Finally, we address the problem of stimating the minimum left extreme, that is, given the $n-$degree stable polynomial $p_0(t)$ and the $(n-1)-$degree polynomial p_1 is such that the vector of coefficients of p_1 satisfies (2) then we find a number k_0 such that $p_0(t) + kp_1(t)$ is Hurwitz for every $k > k_0$. The problem of calculating the minimum left extreme was studied and solved by Bialas with out supposing an extra condition on p_1. Generally, our approach estimates, but does not calculate exactly the minimum left extreme. However the operations are simple and can be applied to $n-$degree polynomials p_1, which is not possible in the result due to Bialas since there it is supposed that $\deg(p_0) > \deg(p_1)$.

The paper is organized as follows: in Section 2 sufficient conditions assuring that a segment of polynomials consists of Hurwitz polynomials are given when it is known that the extremes $p_0(t)$ and $p_1(t)$ are Hurwitz. In Section 3 we suppose that $p_0(t)$ is Hurwitz and we stablish sufficient conditions on $c^T = (c_1, c_2, ..., c_n)$ (the vector of coeficients of $p_1(t) = c_1 t^{n-1} + c_2 t^{n-2} + ... + c_n$) such that $\lambda p_0(t) + (1-\lambda)p_1(t)$ is Hurwitz for all $\lambda \in [0,1]$. Finally in Section 4 we estimate the minimum left extreme.

2 Segments of stable polynomials

The aim of this section is to obtain conditions for the stability of segments of polynomials. We will present the proof of the our results in a forthcoming paper. The main result is based on the following lemma where sufficient conditions for a real polynomial to be Hurwitz are given.

Lemma 1 *Let $F(t)$ and $f(t)$ be real polynomials of degree n, such that $f(0) \neq 0$ and the roots of $F(t)$ are contained in $\mathbb{C}^+$. Consider the $2n-$degree polynomial given by $F(t)f(t)$ and let l be a straight line that passes through the origin. If $F(i\omega)f(i\omega) \neq 0$ and $F(i\omega)f(i\omega)$ does not intersect l for all $\omega > 0$, then all the roots of $f(t)$ are in $\mathbb{C}^+$.*

Remark 2 *Particular cases of lemma 1 are the situations when l is the real axis or the imaginary axis. This cases can easily be used for calculations. In the following theorem we apply the lemma 1 when l is the imaginary axis.*

Theorem 3 *Consider the Hurwitz polynomials $p_0(t) = t^n + a_1 t^{n-1} + ... + a_n$ and $p_1(t) = c_1 t^{n-1} + c_2 t^{n-2} + ... + c_n$. If $c = (c_1, c_2, ..., c_n)^T \succeq 0$ is a solution to (2), then, for all $\lambda \in [0,1]$, the polynomial $\lambda p_0(t) + (1-\lambda)p_1(t)$ is Hurwitz.*

Remark 4 *In the above theorem we have that $\deg(p_0(t)) = n$ and $\deg(p_1(t)) = n-1$. A similar result is obtained if $\deg(p_0(t)) = \deg(p_1(t)) = n$. In that case if $p_0(t) = t^n + a_1 t^{n-1} + ... + a_n$, $p_1(t) = c_1 t^n + c_2 t^{n-1} + ... + c_{n+1}$, the matrix $E_{(n,n)} \in \mathcal{M}_{(n+1)\times(n+1)}$ is defined by*

$$E_{(n,n)} = \begin{pmatrix} 1 & 0 & 0 & 0 & ... & 0 & 0 \\ -a_2 & a_1 & -1 & 0 & ... & 0 & 0 \\ a_4 & -a_3 & a_2 & -a_1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & a_{n-1} & -a_{n-2} \\ 0 & 0 & 0 & 0 & ... & 0 & a_n \end{pmatrix} \quad (3)$$

Remark 5 *There is an obvious relationship between stable rays and stable segments of polynomials: if $p_0(t) + kp_1(t)$ is a stable polynomial for every $k \geq 0$ then $\left(\frac{1}{1+k}\right) p_0(t) + \left(\frac{k}{1+k}\right) p_1(t)$ is a stable polynomial for every $k \geq 0$. This means that the stabilty of the ray $p_0(t) + kp_1(t)$ is equivalent to the stability of the open segment $[p_0(t), p_1(t))$. On the other hand, to get theorem 1 we analyse the real part of complex number $p_0(-i\omega)[\lambda p_0 + (1-\lambda)p_1](i\omega)$, which is given by*

$$p_0(-i\omega)[\lambda p_0 + (1-\lambda)p_1](i\omega) = \lambda \left[P^2(\omega^2) + \omega^2 Q(\omega^2)\right] +$$

ISBN: 970-32-2137-8

$$+(1-\lambda)\left[P(\omega^2)p(\omega^2)+\omega^2 Q(\omega^2)q(\omega^2)\right]+$$
$$+i\omega(1-\lambda)\left[P(\omega^2)q(\omega^2)-Q(\omega^2)p(\omega^2)\right] \quad (4)$$

Other posiblity is to analyse the imaginary part. Such analysis was done in [1] and the results were gotten in terms of rays. Given the Hurwitz polynomial $p_0(t)=t^n+a_1t^{n-1}+...+a_n$ define the matrix

$$D_{(n,n-1)}=\begin{pmatrix} 1 & 0 & 0 & 0 & ... & 0 & 0 \\ -a_2 & a_1 & -1 & 0 & ... & 0 & 0 \\ a_4 & -a_3 & a_2 & -a_1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & a_{n-2} & -a_{n-3} \\ 0 & 0 & 0 & 0 & ... & -a_n & a_{n-1} \end{pmatrix} \quad (5)$$

By the above observations and theorems 1 in [1] if $p_1(t)=c_1t^{n-1}+c_2t^{n-2}+...+c_n$ is Hurwitz and $c=(c_1,c_2,...,c_n)^T\succeq 0$ satisfies the system of linear inequalities

$$D_{(n,n-1)}c\gneqq 0 \quad (6)$$

then, for all $\lambda\in[0,1]$, the polynomial $\lambda p_0(t)+(1-\lambda)p_1(t)$ is Hurwitz.

As was noted in [1] our results can be extended to the cases when the $\deg(p_1(t))=n$ and $n-2$. Condition (6) must be satisfied for a similar matrix D. Given a real polynomial $p_0(t)=t^n+a_1t^{n-1}+...+a_n$ define the matrix $D\in\mathcal{M}_{n\times(n+1)}$ by

$$D_{(n,n)}=\begin{pmatrix} a_1 & -1 & 0 & 0 & ... & 0 & 0 \\ -a_3 & a_2 & -a_1 & 1 & ... & 0 & 0 \\ a_5 & -a_4 & a_3 & -a_2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & -a_{n-2} & a_{n-3} \\ 0 & 0 & 0 & 0 & ... & a_n & -a_{n-1} \end{pmatrix}$$

If $c=(c_1,c_2,...,c_{n+1})^T\succeq 0$ is a solution to $D_{(n,n)}c\gneqq 0$, then $[p_0(t),p_1(t)]$ is a stable segment, where the polynomial $p_1(t)=c_1t^n+c_2t^{n-1}+...+c_{n+1}$ is a Hurwitz polynomial.

For the case when the degree of p_1 is $n-2$, define the matrix $D\in\mathcal{M}_{(n-1)\times(n-1)}$ by

$$D_{(n,n-2)}=\begin{pmatrix} a_1 & -1 & 0 & 0 & ... & 0 & 0 \\ -a_3 & a_2 & -a_1 & 1 & ... & 0 & 0 \\ a_5 & -a_4 & a_3 & -a_2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & a_{n-2} & -a_{n-3} \\ 0 & 0 & 0 & 0 & ... & -a_n & a_{n-1} \end{pmatrix}$$

Then, if $c=(c_1,c_2,...,c_{n-1})^T\succeq 0$ is a solution to $D_{(n,n-2)}c\gneqq 0$, then $[p_0(t),p_1(t)]$ is a stable segment, where the polynomial $p_1(t)=c_1t^{n-2}+c_2t^{n-3}+...+c_{n-1}$ is a Hurwitz polynomial.

Remark 6 *Unlike the results in [1] in this paper we can not obtain a condition when* $\deg(p_0(t))=n$ *and* $\deg(p_1(t))=n-2$ *since the corresponding matrix* $E_{(n,n-2)}$ *is the following matrix in* $\mathcal{M}_{n\times(n-1)}$

$$E_{(n,n-2)}=\begin{pmatrix} -1 & 0 & 0 & 0 & ... & 0 & 0 \\ a_2 & a_1 & -1 & 0 & ... & 0 & 0 \\ -a_4 & -a_3 & a_2 & -a_1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & a_{n-1} & -a_{n-2} \\ 0 & 0 & 0 & 0 & ... & 0 & a_n \end{pmatrix}$$

then the first inequality is $-c_1\geq 0$ *which is not possible since* $c_1>0$.

3 Local Convex Directions

We now study a different problem: a polynomial $p_0(t)$ is given and we know that $p_0(t)$ is Hurwitz, then we wonder if there are polynomials $p_1(t)$ such that $[p_0(t),p_1(t)]$ is a segment of Hurwitz polynomials. This means to find the local convex directions p_1-p_0 with respect to the polynomial $p_0(t)$. The following theorem solves in part this problem.

Theorem 7 *Let* $p_0(t)=t^n+a_1t^{n-1}+...+a_n$ *be a Hurwitz polynomial. Let* $E_{(n,n-1)}$ *be the corresponding matrix defined by (1). If the vector* $c=(c_1,c_2,...,c_n)^T\succeq 0$ *is a solution to the system of linear inequalities (2) then,* $p_1(t)-p_0(t)$ *is a local convex direction with respect to the polynomial* $p_0(t)$*, where* $p_1(t)$ *is given by* $p_1(t)=\sum_{i=1}^n c_it^{n-i}$.

Remark 8 *Note that at the beginning of this section we emphasize that theorem 2 solves the posed problem only "in part". This is because the theorem 2 only gives sufficient conditions on the coefficients of* $p_1(t)$*. Such conditions are the linear inequalities (2), but such inequalities there could not have a solution. Or there could be polynomials* p_1 *such that* $[p_0(t),p_1(t)]$ *is a segment of Hurwitz polynomials, but the vector of coefficients of* $p_1(t)$ *does not satisfy the linear inequality (2) as can be observed in the following example.*

Example 9 *Consider the Hurwitz polynomials* $p_0(t)=t^3+2t^2+t+1$ *and* $p_1(t)=t^2+t+3$,

The matrix $E_{(3,2)}$ *is given by*

$$E_{(3,2)}=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Then the linear inequalities (2) are not satisfied since

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}=\begin{pmatrix} 1 \\ -6 \\ 3 \end{pmatrix}$$

However the segment $[p_0(t),p_1(t)]$ *is a segment of stable polynomials since* $\lambda p_0(t)+(1-\lambda)p_1(t)=$

ISBN: 970-32-2137-8

$\lambda t^3+(\lambda+1)t^2+t+3-2\lambda$ is stable for $\lambda=0$ and for $\lambda\in(0,1]$ the Routh-Hurwitz conditions are satisfied since $\lambda,(\lambda+1),1,3-2\lambda>0$ and $(\lambda+1)(1)-\lambda(3-2\lambda)=2\lambda^2-2\lambda+1>0$ for every $\lambda\in(0,1]$.

This example illustrate the importance of studying the problem of the existence of solutions of (2) and the characteization of the set of solutions.

Definition 10 *It is said that a $m\times s$ real matrix R is a matrix of monotone kind if $Rz\succeq 0$ implies $z\succeq 0$ (see [12]).*

It is known that a square real matrix R is of monotone kind if and only if there exists R^{-1} and $R^{-1}\succeq 0$, where $R^{-1}\succeq 0$ means that all its entries are nonnegative (see [9]).

The matrix $E_{(n,n-1)}$ is of monotone kind, that is it is invertible and $E^{-1}_{(n,n-1)}$ has elements nonnegative. The proof follows similarly as in [1] was proved that the elements of $D^{-1}_{(n,n-1)}$ are nonnegative.

Now, denote by $V=\{z\in R^n/\{0\}\mid z_i\geq 0,\forall i=1,2,...n\}$.

Theorem 11 *The set H of solutions of the system of linear inequalities (2) can be written as $H=E^{-1}_{(n,n-1)}V$.*

Corollary 12 *Let $p_0(t)=t^n+a_1t^{n-1}+...+a_n$ be a Hurwitz polynomial. Let $E_{(n,n-1)}$ be the corresponding matrix defined by (1). If the vector $c=(c_1,c_2,...,c_n)^T\in E^{-1}_{(n,n-1)}V$ then $[p_0(t),p_1(t)]$ is a segment of Hurwitz polynomials, where the polynomial $p_1(t)$ is given by $p_1(t)=\sum_{i=1}^n c_it^{n-i}$.* $\blacksquare$

Remark 13 *If we make the identification of a $n-$degree polynomial $f(t)=b_0t^n+b_1t^{n-1}+...+b_n$ with the vector $(b_0,b_1,...,b_n)\in\mathbb{R}^{n+1}$ and a $(n-1)-$degree polynomial $g(t)=d_1t^{n-1}+d_2t^{n-2}+...+d_n$ with the vector $(0,d_1,...,d_n)\in\mathbb{R}^{n+1}$ then, given the Hurwitz polynomial $p_0(t)$, the polynomials $p_1(t)$ that satisfy (2) can be seen as a polyhedral cone $\mathcal{C}$ generated by $w_1=\left(0,E^{-1}_{(n,n-1)}e_1\right),w_2=\left(0,E^{-1}_{(n,n-1)}e_2\right),...,w_n=\left(0,E^{-1}_{(n,n-1)}e_n\right)$, where $e_1,e_2,...,e_n$ are the canonical vectors in $\mathbb{R}^n$. If we identificate $p_0(t)$ with the vector $w_0=(1,a_1,...,a_n)$ then the local convex directions p_1-p_0 can be seen as vectors $w-w_0$ with $w\in\mathcal{C}$. And the set consisting of the union of all segments $[p_0(t),p_1(t)]$ is the minimal convex set containing $\mathcal{C}$ and w_0.*

Example 14 *Consider the Hurwitz polynomials $p_0(t)=t^3+2t^2+\frac{3}{2}t+1$ and $p_1(t)=2t^2+4t+2$,*

The matrix $E_{(3,2)}$ is given by

$$E_{(3,2)}=\begin{pmatrix}2&-1&0\\-1&\frac{3}{2}&-2\\0&0&1\end{pmatrix}$$

and

$$\begin{pmatrix}2&-1&0\\-1&\frac{3}{2}&-2\\0&0&1\end{pmatrix}\begin{pmatrix}2\\4\\2\end{pmatrix}=\begin{pmatrix}0\\0\\2\end{pmatrix}$$

Consequently $[p_0(t),p_1(t)]$ is a segment of stable polynomials. But this fact can not verified by the results in [1] since the matrix $D_{(3,2)}$ is given by

$$D_{(3,2)}=\begin{pmatrix}1&0&0\\-11&6&-1\\0&-6&11\end{pmatrix}$$

and we have

$$\begin{pmatrix}1&0&0\\-\frac{3}{2}&2&-1\\0&-1&\frac{3}{2}\end{pmatrix}\begin{pmatrix}2\\4\\2\end{pmatrix}=\begin{pmatrix}2\\3\\-1\end{pmatrix}$$

.

On the other hand, we will see that it is not posible to verify the stability of the segment using the Rantzer-type conditions (see [10], [13]).

4 The minimum left extreme

Consider the polynomial

$$p(t,k)=p_0(t)+kp_1(t) \tag{7}$$

where $p_0(t)=t^n+a_1t^{n-1}+...+a_n$ is a Hurwitz stable polynomial and, let $E_{(n,n-1)}$ be the corresponding matrix defined as in (1). If the coefficient vector $c=(c_1,c_2,...,c_n)^T\succeq 0$ of the polynomial $p_1(t)=\sum_{i=1}^n c_it^{n-i}$ is a solution to the system of linear inequalities (2) then, $p_0(t)+kp_1(t)$ is a Hurwitz polynomial $\forall k\geq 0$. For the robust stability analysis of the linear system asociated with $p_0(t)$, we are concerned with estimating the minimum $k_{\min}<0$ such that $p_0(t)+kp_1(t)$ is a Hurwitz polynomial $\forall k\geq k_{\min}$. In [4] it was proved that

$$k_{\min}=\frac{1}{\lambda^-_{\min}[-H^{-1}(p_0)H(p_1)]} \tag{8}$$

where $H(p_0),H(p_1)$ are the corresponding Hurwitz matrices of p_0 and p_1 respectively and, $\lambda^-_{\min}[-H^{-1}(p_0)H(p_1)]$ is the least negative eigenvalue of the matrix $-H^{-1}(p_0)H(p_1)$.

ISBN: 970-32-2137-8

Observe that numerically (8) is not easy to calculate. In the following we get an easy procedure to obtain an estimation of $k_{\min}$. Define the matrix

$$Z_{(n,n-1)} = \begin{pmatrix} a_1 & -2 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_2 & -2a_1 & 2 & \dots & 0 & 0 \\ 0 & 0 & a_3 & -2a_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_{n-1} & -2a_{n-2} \\ 0 & 0 & 0 & 0 & \dots & 0 & a_n \end{pmatrix} \quad (9)$$

and let $Z^i_{(n,n-1)}$ denote the i-th row of matrix $Z_{(n.n-1)}$ and $a = (a_1, a_2, ..., a_n)^T$.

Theorem 15 *Let $p_0(t) = t^n + a_1 t^{n-1} + ... + a_n$ be a Hurwitz polynomial. Let $E_{(n,n-1)}$ be the corresponding matrix defined by (1). If the vector $c = (c_1, c_2, ..., c_n)^T \succeq 0$ is a solution to the system of linear inequalities (2) and the polynomial $p_1(t)$ is given by $p_1(t) = \sum_{i=1}^n c_i t^{n-i}$,*
then $p_0(t)+kp_1(t)$ is a Hurwitz polynomials $\forall k > k_0$, where $k_0 = \max\limits_{i=1,\dots,n} \left(-\frac{Z^i_{(n,n-1)}a}{E^i_{(n,n-1)}c} \right)$, that is $k_{\min} \le k_0$.

Remark 16 *In the same way as we have studied the minimum $k_{\min} < 0$ such that $p_0(t) + kp_1(t)$ is a Hurwitz polynomial $\forall k \ge k_{\min}$ we can study $q_{\min}$, which is the least number to makes stable the family $p_0(t) + q\,[p_1(t) - p_0(t)]$ $\forall q \in (q_{\min}, 1)$. Here the value of the right extreme of $(q_{\min}, 1)$ is 1 because p_0 is an element of the set of Hurwitz polynomials with degree n (that is, $p_0 \in \mathcal{H}_n$) and for $q = 1$ we obtained p_1 which is a polynomial on the boundary of $\mathcal{H}_n$. Let $p_0(t) = t^n + a_1 t^{n-1} + ... + a_n$ be a Hurwitz polynomial and let $E_{(n,n-1)}$ be the corresponding matrix defined by (1). If the vector $c = (c_1, c_2, ..., c_n)^T \succeq 0$ is a solution to the system of linear inequalities $Ec \succneqq 0$ and the polynomial $p_1(t)$ is given by $p_1(t) = \sum_{i=1}^n c_i t^{n-i}$, define $k_0 = \max\limits_{i=1,\dots,n} \left(-\frac{Z^i a}{E^i c} \right)$. Consequently $p_0(t) + kp_1(t)$ is Hurwitz $\forall k > k_0$. If $k_0 > -1$ then $\frac{1}{1+k}p_0(t) + \frac{k}{1+k}p_1(t) = p_0(t) + \frac{k}{1+k}(p_1(t) - p_0(t))$ is Hurwitz $\forall k > k_0$ and then $q_{\min}$ can be estimated by $q_0 = \frac{k_0}{1+k_0}$. If $k_0 \le -1$, consider the limit of $\frac{k}{k+1}$ when $k \to -1^+$, we get that $\frac{k}{k+1} \to -\infty$ and then it is satisfied that $p_0(t) + q(p_1(t) - p_0(t))$ is Hurwitz $\forall k \in (-\infty, 1)$* ■

Example 17 *Given the polynomials $p_0(t) = t^3 + 7t^2 + 14t + 8$, $p_1(t) = t^2 + 4t + 6$*

$$Z_{(3,2)} = \begin{pmatrix} 7 & -2 & 0 \\ 0 & 14 & -14 \\ 0 & 0 & 8 \end{pmatrix}, \; a = \begin{pmatrix} 7 \\ 14 \\ 8 \end{pmatrix},$$

$$Z_{(3,2)}a = \begin{pmatrix} 21 \\ 84 \\ 64 \end{pmatrix}$$

$$E_{(3,2)} = \begin{pmatrix} 7 & -1 & 0 \\ -8 & 14 & -7 \\ 0 & 0 & 8 \end{pmatrix}, \; c = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix},$$

$$E_{(3,2)}c = \begin{pmatrix} 3 \\ 6 \\ 48 \end{pmatrix}$$

$$k_0 = \max\left(-\tfrac{21}{3}, -\tfrac{84}{6}, -\tfrac{64}{48}\right) = -\tfrac{4}{3} < -1$$

$$H(p_0) = \begin{pmatrix} 7 & 8 & 0 \\ 1 & 14 & 0 \\ 0 & 7 & 8 \end{pmatrix}, \; H(p_1) = \begin{pmatrix} 1 & 6 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 6 \end{pmatrix}$$

$$H^{-1}(p_0)H(p_1) = \begin{pmatrix} \frac{7}{45} & \frac{26}{45} & 0 \\ -\frac{1}{90} & \frac{11}{45} & 0 \\ \frac{7}{720} & -\frac{4}{45} & \frac{3}{4} \end{pmatrix},$$

$\sigma\left(-H^{-1}(p_0)H(p_1)\right) = \{-\frac{3}{4}, -\frac{1}{5} \pm \frac{1}{15}i\}$

$\lambda_{\min} = -\frac{3}{4}$, $k_{\min} = -\frac{4}{3} < -1$. *Then, in this example the two approaches lead to the conclusion that $p_0(t) + kp_1(t)$ is Hurwitz $\forall k > -\frac{4}{3}$ and $p_0(t) + q\,[p_1(t) - p_0(t)]$ is Hurwitz $\forall q \in (-\infty, 1)$.*

Example 18 $p_0(t) = t^3 + 7t^2 + 14t + 8$, $p_1(t) = 26t^2 + 137t + 90$

$$Z_{(3,2)} = \begin{pmatrix} 7 & -2 & 0 \\ 0 & 14 & -14 \\ 0 & 0 & 8 \end{pmatrix}, \; a = \begin{pmatrix} 7 \\ 14 \\ 8 \end{pmatrix},$$

$$Z_{(3,2)}a = \begin{pmatrix} 21 \\ 84 \\ 64 \end{pmatrix}$$

$$E_{(3,2)} = \begin{pmatrix} 7 & -1 & 0 \\ -8 & 14 & -7 \\ 0 & 0 & 8 \end{pmatrix}, \; c = \begin{pmatrix} 26 \\ 137 \\ 90 \end{pmatrix},$$

$$E_{(3,2)}c = \begin{pmatrix} 45 \\ 1080 \\ 720 \end{pmatrix}$$

$$k_0 = \max\left(-\tfrac{21}{45}, -\tfrac{84}{1080}, -\tfrac{64}{720}\right) = -\tfrac{7}{90} = -0.07778 > -1$$

$$H(p_0) = \begin{pmatrix} 7 & 8 & 0 \\ 1 & 14 & 0 \\ 0 & 7 & 8 \end{pmatrix}, \; H(p_1) = \begin{pmatrix} 26 & 90 & 0 \\ 0 & 137 & 0 \\ 0 & 26 & 90 \end{pmatrix}$$

$$H^{-1}(p_0)H(p_1) = \begin{pmatrix} \frac{182}{45} & \frac{82}{45} & 0 \\ -\frac{13}{45} & \frac{869}{90} & 0 \\ \frac{91}{360} & -\frac{3743}{720} & \frac{45}{4} \end{pmatrix},$$

$\sigma\left(-H^{-1}(p_0)H(p_1)\right) = \{-11.25, -4.1399, -9.5601\}$

$\lambda_{\min} = -11.25$, $k_{\min} = -0.088889 > -1$. *Then, in this example our approach leads to*

$p_0(t) + kp_1(t)$ is Hurwitz $\forall k > -0.07778$ and

$p_0(t) + q\,[p_1(t) - p_0(t)]$ is Hurwitz $\forall q \in (-0.08434, 1)$

and the approach of Bialas leads to

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$p_0(t) + kp_1(t)$ is Hurwitz $\forall k > -0.088889$ and
$p_0(t) + q[p_1(t) - p_0(t)]$ is Hurwitz $\forall q \in (-0.097561, 1)$.

Remark 19 *An advantage of our approach compared with that of Bialas is that we can estimate the minimum left extremum when the two polinomials have the same degree. Let $p_0(t) = t^n + a_1 t^{n-1} + ... + a_n$. The corresponding matrix $Z_{(n,n)}$ is defined by*

$$Z_{(n,n)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_1 & -2 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_2 & -2a_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_{n-1} & -2a_{n-2} \\ 0 & 0 & 0 & 0 & \dots & 0 & a_n \end{pmatrix}$$

Let $E_{(n,n)}$ be the corresponding matrix defined by (3). If the vector $c = (c_0, c_1, ..., c_n)^T \succeq 0$ is a solution to the system of linear inequalities $E_{(n,n)}c \gneqq 0$ and the polynomial $p_1(t)$ is given by $p_1(t) = \sum_{i=0}^{n} c_i t^{n-i}$, then $p_0(t) + kp_1(t)$ is a Hurwitz polynomial $\forall k > k_0$, where $k_0 = \max\limits_{i=1,\dots,n,n+1} \left(-\frac{Z^i a}{E^i c}\right)$. That is $k_{\min} \leq k_0$.

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